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**SIMILAR SOLUTIONS IN VIBRATIONAL NONEQUILIBRIUM
NOZZLE FLOWS**

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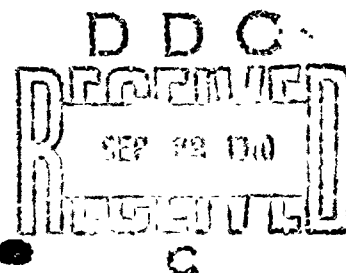
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FLOWS**

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**AIAA 3rd Fluid and Plasma
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SIMILAR SOLUTIONS IN VIBRATIONAL NONEQUILIBRIUM NOZZLE FLOWS

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ABSTRACT

The problem of obtaining similar solutions in pseudo-one-dimensional nozzle flows with vibrational relaxation is considered. The governing equations are transformed into a similar form by using a new similarity parameter η so that the nonequilibrium flow solutions depend on two parameters for a given kind of gas. However, the similar equations are further reduced to a universal form by a transformation of the independent variable η to ξ such that the similar solutions, for all combinations of initial conditions and for a defined family of nozzle shapes, depend on a single parameter χ which are then presented in a single graph. It is also shown that the equilibrium solutions depend on only one parameter ξ . The parameters ξ and χ are also the exact, general correlating parameters. With the present similar solutions the flow quantities in a nozzle are readily determined without resorting to complex computer programs.

NOMENCLATURE

A = area ratio (A'/A'_*)
 a = speed of sound
 C = constant defined in Eq (8)
 D = constant defined in Eq (8)
 f_1, f_2, f_3 = functions defined in Appendix A
 H_0 = reservoir enthalpy ($H_0 = H'_0/u_d^2$)
 h = static enthalpy ($h = h'/u_d^2$)
 k_1, k_2 = functions defined in Appendix B
 L = nozzle scale parameter ($L = r'_*/\tan \gamma$)
 M = flow Mach number
 m = molecular weight
 N = number of vibrational levels ($N = T_d/\theta_v$)
 $(N_s)_2$ = defined in Eq (17)
 $(N_s)_1$ = defined in Eq (26)
 N_s = defined in Eq (31)
 p_d = characteristic dissociation pressure ($p_d = p_d u_d^2$)

p = pressure ($p = p'/p_d$)
 R = molecular gas constant ($R = R/m$)
 r'_* = nozzle throat radius
 S_0 = reservoir entropy ($S_0 = S'_0/R$)
 S_r = reservoir entropy ($S_r = S'_r/R$) = 15.2 for nitrogen
 T_d = characteristic dissociation temperature
 T_t = translational temperature ($T_t = T'_t/T_d$)
 T_v = vibrational temperature ($T_v = T'_v/T_d$)
 u_d = characteristic dissociation velocity [$u_d = (\lambda T_d)^{1/2}$]
 u = velocity ($u = u'/u_d$)
 u'_0 = velocity defined as $(p'_0/\rho'_0)^{1/2}$
 x = distance along the nozzle axis ($x = x'/L$)
 ϵ = vibrational energy ($\epsilon = \epsilon'/RT_d$)
 $\epsilon(\infty)$ = equilibrium vibrational energy
 τ_v = vibrational relaxation time constant
 η = function defined as $\log_e(uA/\rho_* u_*)$
 ρ_d = characteristic dissociation density
 ρ = density ($\rho = \rho'/\rho_d$)
 θ = characteristic vibrational temperature ($\theta = \theta_v/T_d$)
 2γ = expansion angle of conical nozzle or asymptotic cone angle of hyperbolic nozzle
 $\lambda_2, \lambda_1, \lambda$ = parameters defined in Eqs (17) (27) and (34) respectively
 ϕ = vibrational temperature function defined as $\phi = (\theta_v/T'_v)$
 ψ = translational temperature function defined as $\psi = (\theta_v/T'_t)$

X_2, X_1, X = parameters defined in Eqs (24), (28) and (35) respectively

ξ = $(S_0 - \eta)$

Subscripts

* = nozzle throat

0 = reservoir

Superscript

' = dimensional quantity

I. INTRODUCTION

The phenomenon of thermal and/or chemical nonequilibrium in flowing gases at high temperatures is of fundamental importance in rocket nozzles as well as in laboratory facilities which are being used to simulate the flow fields over bodies in hypersonic flight. In the past decade a considerable amount of theoretical effort has been directed toward understanding the nonequilibrium flow effects in nozzles under steady flow conditions. The problem has been studied under the assumption of pseudo-one-dimensional, adiabatic, inviscid flow. In spite of these simplifying assumptions the solutions are far from being simple and are often plagued by many numerical procedural difficulties. A comprehensive review of this problem is presented in Ref 1. More recently, a time-dependent analysis² has been proposed which circumvents some of the numerical difficulties but retains the problem of determining the flow quantities through a numerous-stepped process. The basic problem of obtaining numerical solutions for nozzle flows with vibrational energy relaxation has been studied by several authors (e.g., Refs 3, 4, 5). The present state-of-the-art for solving vibrational nonequilibrium nozzle flows requires complex computer programs with which the flow variables are determined by numerical integration for any given initial and boundary conditions. However, this approach does not provide suitable theoretical comparisons for use by the experimentalist because of the many variables involved. Thus, it is apparent that general correlating parameters are needed. Several approximate analyses^{6,7} have been used to predict the frozen vibrational temperature in nozzle flows. In these analyses approximate correlating parameters have been deduced which do not include all the variables of the problem. Furthermore, these analyses predict only the frozen vibrational temperature.

From the preceeding discussion it is obvious that suitable similar solutions to this problem are highly desirable. Such similar solutions would not only eliminate the need for repeated complex computations but also provide for the experimentalist, the badly needed correlating parameters.

In the present analysis, the governing equations, for a pseudo-one-dimensional nonequilibrium nozzle flow with vibrational energy relaxation but no dissociation, are transformed into a similar form by using a new independent variable η . The definition of the similarity parameter η and the method of transformation of the governing equations in the present case are very similar to those used in the analysis presented by the authors⁸ which deals with the problem of dissociational nonequilibrium nozzle flows. It is shown that the similar solutions, for a family of nozzle shapes and a specified gas, depend on two parameters, S_0 and λ , in addition to the independent variable η . However, the similar equations are further reduced to a universal form by a transformation of the independent variable η to ξ so that the similar solutions depend on a single parameter χ with ξ as the independent variable. General similar solutions which can be used for all combinations of initial conditions are presented in a single graph for nitrogen. The parameters χ and ξ are the exact and general correlating parameters. The limiting solutions, namely frozen and equilibrium solutions, are also obtained with ξ as the independent variable and these solutions depend on ξ only. The approximate correlating parameters that have appeared in the literature^{6,7} can be deduced from the present general correlating parameter χ and the approximations involved in these analyses are discussed.

II. TRANSFORMATION OF GOVERNING EQUATIONS

In the analysis of the dissociation nonequilibrium case⁸ it was logical to use the characteristic dissociation values to nondimensionalize the governing equations since the characteristic dissociation density and temperature appear in the governing equations. The same values are also used for nondimensionalization in the present case for the following reasons. The governing equations contain only the characteristic vibrational temperature θ_v and there is no corresponding characteristic density. Furthermore, it is shown later that the similar solutions in the present case do not depend on the values used for nondimensionalization and hence can be arbitrarily chosen.

The governing equations for a steady, pseudo-one-dimensional, adiabatic, inviscid flow with negligible dissociation are considered. The flow variables p' , ρ' , T' and u' are nondimensionalized with the corresponding dissociation values which are defined in the nomenclature. The nozzle area and the distance along the nozzle axis are nondimensionalized by the nozzle throat area A^* and scale parameter L , respectively. The governing equations after nondimensionalization are written as

Equation of state:

$$p = \rho T_t \quad (1)$$

Conservation of mass:

$$\rho u A = \rho_* u_* \quad (2)$$

Conservation of energy:

$$u^2/2 + h = H_0 = \text{const} \quad (3)$$

where

$$h = (7/2)T_t + \varepsilon$$

Conservation of momentum:

$$u du + dp/\rho = 0 \quad (4)$$

Rate equation (Landau-Teller type):

$$\frac{d\varepsilon}{dx} = \frac{L}{uu_d \tau_v} [\varepsilon(\infty) - \varepsilon] \quad (5)$$

where $\varepsilon(\infty)$ corresponds to vibrational equilibrium. Applying the cut-off harmonic oscillator approximation (see Appendix A), the vibrational energy ε can be given in terms of vibrational temperature as

$$\varepsilon = \frac{\theta}{e^{\theta/T_v} - 1} - \frac{N\theta}{e^{N\theta/T_v} - 1} \quad (6)$$

where a Boltzmann distribution of the molecular number density in the different vibrational levels is assumed. The corresponding vibrational energy at vibrational equilibrium $\varepsilon(\infty)$ is given as

$$\varepsilon(\infty) = \frac{\theta}{e^{\theta/T_t} - 1} - \frac{N\theta}{e^{N\theta/T_t} - 1} \quad (7)$$

The vibrational relaxation time constant τ_v has been correlated³ for many gas systems by using measured values and may be represented by an expression of the form

$$\tau_v p' = C e^{D(\theta_v/T_t')^{1/3}} \quad (8)$$

where C and D are constants for a given gas.

An independent variable η is now defined as $\eta = \log_e(uA/\rho_* u_*)$ and the temperature functions are defined as $\phi = \theta/T_v \equiv \theta_v/T_v'$ and $\psi = \theta/T_t \equiv \theta_v/T_t'$. Using these definitions and Eqs (7) and (8), the governing equations are reduced to

Equation of state:

$$p_v = \theta e^{-\eta} \quad (9)$$

Conservation of mass:

$$\rho = e^{-\eta} \quad (10)$$

Conservation of energy:

$$u^2 + \theta \left[\frac{7}{2} \frac{1}{\psi} + \frac{1}{e^{\phi} - 1} - \frac{N}{e^{N\phi} - 1} \right] = H_0 \quad (11)$$

Conservation of momentum:

$$d\eta - (5/2)(d\psi/\psi) - (1-f_1)\psi e^{\phi} d\phi / (e^{\phi} - 1)^2 = 0 \quad (12)$$

Rate equation:

$$\frac{d\phi}{dx} = \frac{L\theta p_d e^{-\eta} e^{-D\psi}^{1/3}}{C u u_d \psi e^{\phi}} \left(\frac{e^{\phi} - 1}{e^{\psi} - 1} \right) (e^{\psi} - e^{\phi}) \frac{(1-f_2)}{(1-f_1)} \quad (13)$$

where f_1 and f_2 are the factors which take into account the effect of the cut-off harmonic oscillator approximation and are given in Appendix A. It is shown in Appendix A that these f functions tend to zero for $N \rightarrow \infty$, corresponding to the simple harmonic oscillator approximation. Furthermore, it is also shown in Appendix A that, for temperatures even as high as 7000°K for nitrogen, these functions are negligible. Hence, the simple harmonic oscillator approximation is sufficiently accurate for the present analysis.

The independent variable x in Eq (13) is changed to η by using the following procedure. From the conservation of momentum and mass the following relation can be derived:

$$d \log_e \rho / d \log_e A \equiv -(d\eta / d \log_e A) = M^2 / (1 - M^2) \quad (14)$$

where $M = u/a$ and $a^2 = dp/d\rho$. For a family of nozzles with the area distribution given by $A = (1 + x^j)^{1/j}$ and using Eq (14) the following expression can be written:

$$\frac{d\eta}{dx} = ij \left(\frac{M^2}{M^2 - 1} \right) A^{-1/ij} (1 - A^{-1/ij})^{(j-1)/j} \quad (15)$$

Substituting for $A^{-1/ij}$ from the definition of η , Eq (15) can be written as

$$\frac{d\eta}{dx} = ij \left(\frac{M^2}{M^2 - 1} \right) (\rho_* u_*)^{-1/ij} \times u^{1/ij} e^{-\eta/ij} (1 - A^{-1/ij})^{(j-1)/j} \quad (16)$$

Combining Eq (13) and Eq (16) the rate equation is transformed to

$$\frac{d\psi}{d\eta} = \frac{1}{(N_s)_\psi} e^{\lambda_2 \psi - 1} e^{-[\eta(1-1/ij) + \psi]^{1/3} D + \phi} \times \left(\frac{e^{\phi} - 1}{e^{\psi} - 1} \right) (e^{\psi} - e^{\phi}) \left(\frac{1-f_2}{1-f_1} \right) \quad (17)$$

where

$$\lambda_2 = \log_e \left[(\rho_* u_*)^{1/ij} L \rho_d u_d \theta / ij C \right]$$

and

$$(N_s)_2 = \left(\frac{M^2}{M^2 - 1} \right) (1 - A^{-1/ij})^{(j-1)/j} u^{(1+1/ij)}$$

The problem under consideration has been reduced to solving two differential equations, Eqs (12) and (17), for two unknowns ψ and ϕ with η as the independent variable. Once ψ and ϕ are determined by solving Eqs (12) and (17), the other unknowns, p , ρ and u are obtained from the other governing equations which are simple algebraic equations.

Character of Similar Equations

The main motivation in expressing the rate equation in the form shown in Eq (17) is to combine all the parameters of the problem into a single parameter λ_2 . However, Eq (17) also contains the additional parameters D and ij . The parameter D is a constant for a given gas and this means separate solutions have to be obtained for each kind of gas. The parameter ij is the nozzle shape parameter and a given value of ij covers a family of nozzle shapes; its value is 2.0 for nozzle shapes of practical interest, namely conical ($i = 2$, $j = 1$) and hyperbolic ($i = 1$, $j = 2$) shapes. Equation (17) also contains a rather undesirable term $(N_s)_2$ which is a function of the flow quantities in the nozzle and is not a constant. In this respect this term may be called a nonsimilar function. The properties of $(N_s)_2$ and a method of including the effect of $(N_s)_2$ into the solutions of similar Eqs (12) and (17) are discussed in one of the following sections. In addition to specifying λ_2 , D and ij , the initial values of ψ , ϕ , for a given value of η , have to be specified for the solutions of Eqs (12) and (17). The specification of these initial values will be facilitated by the limiting solutions, namely the frozen and equilibrium solutions.

III. LIMITING SOLUTIONS

Frozen flow Solution

This limiting case is achieved when the vibrational relaxation time τ_v is very large and this happens if $\lambda_2 \rightarrow \infty$. Then $d\phi/d\eta \rightarrow 0$ follows from the rate equation and hence the trivial solution $\phi = \text{constant}$. The generalized momentum equation (Eq (12)) simplifies to

$$d\eta - (5/2)(d\psi/\psi) = 0 \quad (18)$$

This equation can be integrated and given as

$$\rho = (\text{const})(\bar{T})^{5/2} \equiv (\text{const})T^{[1/(\gamma-1)]} \quad (19)$$

for $\gamma = 7/5$. Equation (19) is the familiar isentropic relation for a perfect gas.

Equilibrium Flow Solution

This limiting case is achieved when the vibrational relaxation time τ_v is very short and this happens if $\lambda_2 \rightarrow 0$. With this situation the condition of $\phi = \psi$ can be inferred from the rate equation (Eq (17)). With $\phi = \psi$, the generalized momentum equation (Eq (12)) can be integrated and given as

$$\eta - (5/2) \log_e \psi - \log_e (1 - e^{-\psi}) + \psi/(e^{\psi} - 1) + f_3 = \text{const} \quad (20)$$

where the function f_3 is a term associated with the cut-off harmonic oscillator approximation and is negligible as shown in Appendix A. Equation (20) is the expression for the change in entropy in an equilibrium flow.¹⁰ This equation also shows that entropy is conserved. The expression for entropy in this case can be given as

$$S_0 \equiv (S'_0/R) = \eta - (5/2) \log_e \psi - \log_e (1 - e^{-\psi}) + \frac{\psi}{e^{\psi} - 1} + f_3 + S_r \quad (21)$$

where $S_r = S'_r/R$ and is the reference entropy. The equilibrium flow solution can be obtained from Eq (21) with η as the independent variable and S_0 as the parameter. It is worth noting that the limiting solutions do not depend on the nozzle geometry.

IV. NONEQUILIBRIUM SOLUTIONS

Initial Values

To obtain nonequilibrium solutions, a set of initial flow quantities ψ , ϕ and η have to be specified before solving the similar governing Eqs (12) and (17). If the flow starts with frozen initial conditions it is known to remain in the frozen state. Hence, the solution is obtained from the simple algebraic equations corresponding to the frozen flow case. If the flow starts with an initial state which is in nonequilibrium, the similar governing equations (Eqs (12) and (17)) have to be solved using the specified initial values of ψ , ϕ and η for the given parameters λ_2 , D and ij . However, in almost all practical situations of nozzle flows, the flow starts in the nozzle reservoir with equilibrium conditions and remains in near-equilibrium up to the nozzle throat.^{1,8} This will become evident when the variation of $(N_s)_1$ along the nozzle axis is discussed in the following section. Therefore, in the present analysis the flow is assumed to be in an equilibrium condition up to the throat. In this case the function $\phi = \psi$

and the initial value of ψ can be obtained from the equilibrium solution (Eq (21)) for values of η and S_0 . In this rather indirect way the entropy S_0 also appears as a parameter in the nonequilibrium solutions. It is noted that this behavior is very similar to that of the dissociational nonequilibrium case.⁸

For a given gas (D constant) and a family of nozzle shapes (ij constant) the parameters λ_2 and S_0 have to be specified for the nonequilibrium solutions. However, this two parametric dependence can be reduced to a single one by the following transformation. A new variable ξ is defined as $\xi = (S_0 - \eta)$. Then Eq (21), which gives the equilibrium solutions, reduces to

$$\xi = \frac{\psi}{e^{\psi-1}} - \log_e [\psi^{5/2} (1-e^{-\psi})] + f_3 + S_r \quad (22)$$

Thus, the equilibrium solutions may be represented by a single universal curve showing the variation of ψ with the independent variable ξ . The governing similar equations (Eq (12) and (17)) for the nonequilibrium case reduce to

$$(5/2)d(\log_e \psi) + d\xi + \psi [e^{\psi}/(e^{\psi}-1)^2] d\phi(1-f_1) = 0 \quad (23)$$

$$\frac{d\phi}{d\xi} = \frac{1}{(N_s)_2} e^{X_2} \psi^{-1} e^{[(1-1/ij)\xi - D\psi^{1/3} - \phi]} \times \left(\frac{e^{\phi}-1}{e^{\psi}-1} \right) (e^{\phi}-e^{\psi}) \left(\frac{1-f_2}{1-f_1} \right) \quad (24)$$

where

$$X_2 = [\lambda_2 - (1-1/ij)S_0]$$

The nonequilibrium solutions depend on a single parameter X_2 for a given gas and for a family of nozzle shapes. The initial values for the nonequilibrium solutions can be easily obtained from Eq (22) for equilibrium starting conditions. The parameter X_2 contains the effects of the vibrational relaxation time constant τ_v , the reservoir conditions and the nozzle shape. The nonsimilar function $(N_s)_2$ has to be taken into account in order to obtain an exact nonequilibrium solution from Eqs (23) and (24). The properties of $(N_s)_2$ and a method of its correlation are discussed in the following section.

Nonsimilar Function N_s

The function $(N_s)_2$ given in Eq (24) is a function of M , u and A . Hence, it will have different values for different reservoir conditions and it also varies along the nozzle axis. The velocity u in the expression for $(N_s)_2$ has been nondimensionalized by u_d which is independent of the

reservoir conditions. It was observed from a number of nozzle computations that the velocity, when nondimensionalized with a velocity of $u'_0 = (p'_0/\rho'_0)^{1/2}$, does not change very much for different reservoir conditions. Therefore, the expression for $(N_s)_2$ is rewritten as

$$(N_s)_2 = \left(\frac{M^2}{M^2-1} \right) (1-A^{-1/ij})^{(j-1)/j} u_1^{(1+1/ij)} \times \left(\frac{u'_0}{u_d} \right)^{(1+1/ij)} \quad (25)$$

where $u_1 = (u'/u'_0)$. The constant term $u'_0/u_d^{(1+1/ij)}$ in Eq (25) can be included in the expression for λ_2 in Eq (17) and the remaining terms can be taken as the nonsimilar function $(N_s)_1$, namely:

$$(N_s)_1 = \left(\frac{M^2}{M^2-1} \right) (1-A^{-1/ij})^{(j-1)/j} u_1^{(1+1/ij)} \quad (26)$$

and

$$\lambda_1 = \lambda_2 + \log_e (u_d/u'_0)^{(1+1/ij)} \quad (27)$$

Then the parameter X_2 defined in Eq (26) becomes

$$X_1 = [\lambda_1 - (1-1/ij)S_0] \quad (28)$$

By letting $i = 1.0$ and $j = 2.0$ (corresponding to the hyperbolic nozzle case) Eq (26) is rewritten as:

$$(N_s)_1 = \pm \left(\frac{M^2}{M^2-1} \right) (1-A^{-1}) u_1^{3/2} \quad (29)$$

A typical variation of $(N_s)_1$ with area ratio is shown in Fig 1. It is noted that moving upstream from the nozzle throat, $(N_s)_1$ rapidly tends to zero since M and u_1 both $\rightarrow 0$. At the geometric throat the rate of change of area with x goes to zero for all nozzles with no area discontinuity at

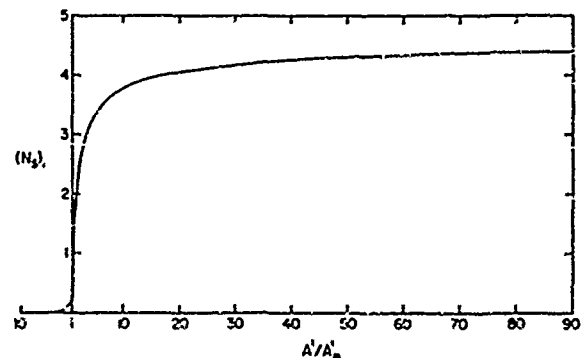


Figure 1. Typical Variation of Function $(N_s)_1$ with Area Ratio for Nitrogen

the throat. Therefore, $(N_g)_1$ has an indeterminate form since M also becomes unity. However, it can be shown that it tends to a definite limit at the throat. To obtain this limit, the complete expression for $(d\eta/dx)$ given below must be considered.

$$\frac{d\eta}{dx} = \frac{1}{A} \frac{dA}{dx} \left(\frac{M^2}{M^2 - 1} \right) \quad (30)$$

If an area discontinuity exists at the throat (for example at the juncture of two conical nozzles) the rate of change of area has a finite value and $(d\eta/dx)$ tends to infinity at the throat. However, nozzles used in practice generally have no area discontinuity at the throat so that the function $(N_g)_1$ is expected to behave as shown in Fig 1. The function $(N_g)_1$ remains positive even for $M < 1.0$ in the upstream portion of the nozzle since dA/dx is negative and hence the negative sign in Eq (29) applies to the upstream portion from the nozzle throat.

Vibrational nonequilibrium nozzle flow quantities, starting at the nozzle throat, were computed for nitrogen for a number of reservoir conditions by using the computer program of Ref 11. The ξ values were also computed using the equation $\xi = (S_0 - \eta) \equiv (S_0 + \log_e \rho)$. The reservoir entropy was computed from Eq (21) where the reference entropy S_r was taken equal to 15.2 for nitrogen which makes the entropy values the same as those computed in Ref 12. The $(N_g)_1$ values were also computed for several reservoir conditions, using Eq (26), and are plotted in Fig 2 with (ξ_*/ξ) as the variable. Although all the curves correlate fairly well immediately downstream of the nozzle throat, a significant temperature effect shows up for ξ_*/ξ values greater than about 1.2. In an effort to obtain a better correlation the following approach was taken. In the mass flow correlation analysis (Appendix B) it is noticed that the nondimensional mass flow $(\rho_*' u_*' / \rho_0' u_0')$ is also slightly temperature dependent. Therefore, the $(N_g)_1$ values were multiplied by

the corresponding mass flow ratio as given below

$$N_g = (N_g)_1 [\rho_*' u_*' / \rho_0' u_0']^{6.0} \quad (31)$$

where the exponent 6.0 was determined by equating the N_g values corresponding to the maximum and minimum values of $(N_g)_1$ given in Fig 2. The N_g values computed from Eq (31) are shown in Fig 3 for the same reservoir conditions used in Fig 2. All the values for different reservoir conditions correlate very well and can be represented by a mean curve as shown by the dotted line in Fig 3. The maximum discrepancy of the actual values from the mean curve is within a few percent. The mean curve can be also represented by a simple analytical equation of the form:

$$N_g = 0.37 - 0.32 (2.0 - \xi_*/\xi)^{6.6} \quad (32)$$

for

$$1 < (\xi_*/\xi) < 2.0$$

and

$$N_g = 0.37 \text{ for } (\xi_*/\xi) > 2.0$$

The N_g values were also computed with different hyperbolic nozzle shapes ($L = 0.5$ to 2.0) as well as combined hyperbolic ($L = 0.5$ to 2.0) and conical ($L = 1$ to 3) shapes. It was found that the differences in the N_g values were within the accuracy of the correlation shown in Fig 3. The factor $[\rho_*' u_*' / \rho_0' u_0']^{6.0}$ was also included in the λ_1 expression so that its effect is properly taken into account. After including the mass flow factor, the final expressions for N_g , λ and χ are,

$$N_g = \left(\frac{M^2}{M^2 - 1} \right) (1 - A^{-1/1})^{(j-1)/j} \times u_1^{(1+1/1j)} \left[\frac{\rho_*' u_*'}{\rho_0' u_0'} \right]^{6.0} \quad (33)$$

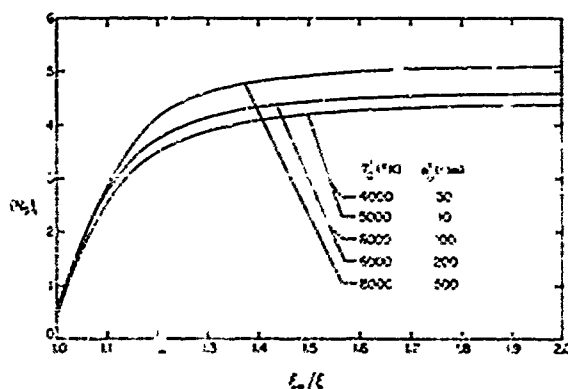


Figure 2. Variation of Function $(N_g)_1$ with the Parameter (ξ_*/ξ) for Nitrogen

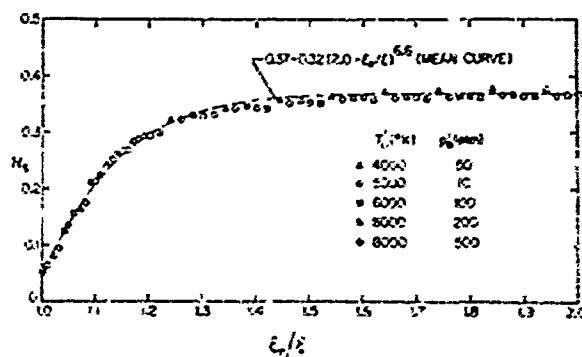


Figure 3. Correlation of Function N_g with the Parameter (ξ_*/ξ) for Nitrogen

$$\lambda = \log_e \left[\frac{L \rho_d u_d \theta}{i j \bar{c}} \left(\frac{\rho_*' u_*'}{\rho_d u_d} \right)^{1/i j} \right] \times$$

$$\left(\frac{u_d}{u_0} \right)^{1+1/i j} \left(\frac{\rho_*' u_*'}{\rho_d u_d} \right)^{6.0} \quad (34)$$

$$\chi = [\lambda - (1-1/i j) S_0] \quad (35)$$

Discussion of Nonequilibrium Solutions

The two similar governing equations (Eqs (23) and (24) with N_8 given by Eq (32) were solved by a fourth-order Runge-Kutta technique with $i j = 2.0$ and $D = 14.7$ which corresponds to nitrogen.⁹ The starting values of ψ and ϕ were obtained for a given ξ from the equilibrium solution ($\psi = \phi$) given by Eq (22). The reference entropy S_r in Eq (22) was taken equal to 15.2. The factors f_1 , f_2 , and f_3 are all assumed to be zero (see Appendix A). The starting values of ξ for different χ values were selected in such a way that the solution always starts with equilibrium conditions. The equilibrium solution obtained from Eq (22) is shown in Fig 4 and is represented by a single universal curve since it is independent of χ , $i j$ and D . Also, in Fig 4 a typical similar nonequilibrium solution is compared with an exact solution obtained from the computer program of Ref 11. The comparison is considered to be very good.

A series of solutions for different values of χ for nitrogen ($D = 14.7$) are shown in Fig 5. The vibrational temperature function ϕ is seen to follow the translational temperature function ψ very closely for awhile, the extent of which depends on χ , and then diverges rather suddenly and reaches a constant value; this corresponds to the freezing of the vibrational energy mode. The translational temperature

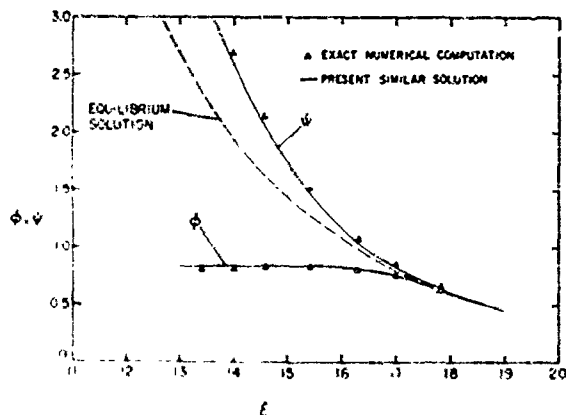


Figure 4. Comparison of Present Similar Solutions with an Exact Solution ($\chi = 4.4$, $D = 14.7$, $i j = 2.0$)

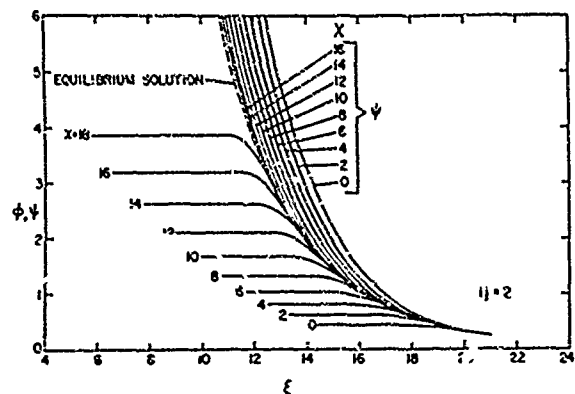


Figure 5. Similar Solutions for Vibrational Nonequilibrium Nitrogen Flow ($D = 14.7$)

function ψ increases monotonically as ξ decreases. The equilibrium solution shown in Fig 5 is also given by the envelope of all the nonequilibrium solutions. The constant $D = 14.7$ for nitrogen was obtained from the correlation of experimental values over a certain temperature range.⁹ However, D can have slightly different values for the same gas over different ranges of temperatures. Therefore, similar solutions were also computed with $D = 13.5$ and 15.5 and are shown in Figs 6 and 7 respectively. It would be an easy matter to interpolate between these solutions for slightly different values of D . The frozen vibrational temperature function ϕ_f depends on χ only and the variation of ϕ_f with χ for all the three values of D , is shown in Fig 8.

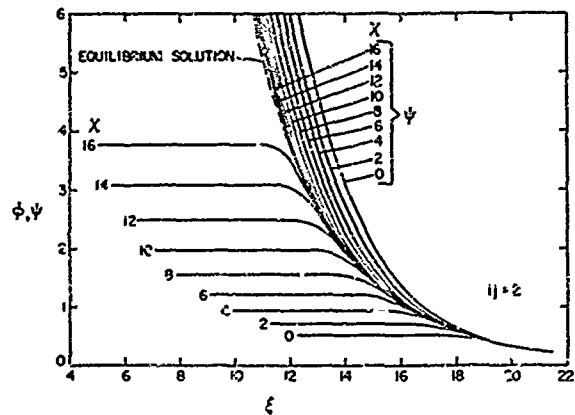


Figure 6. Similar Solutions for Vibrational Nonequilibrium Nitrogen Flow ($D = 13.5$)

V. THE PARAMETERS ξ AND χ

It is shown in this analysis that the nonequilibrium similar solutions depend on two general parameters ξ and χ . In order to use the similar solutions presented in this report the parameters ξ and χ should be known in terms of the initial and boundary values. Therefore, the functional dependence of ξ and χ are considered in this section.

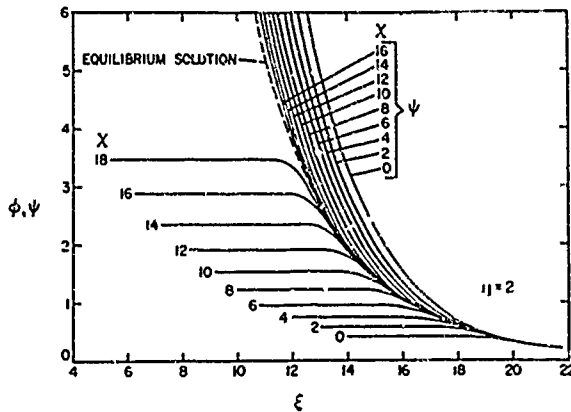


Figure 7. Similar Solutions for Vibrational Nonequilibrium Nitrogen Flow ($D = 15.5$)

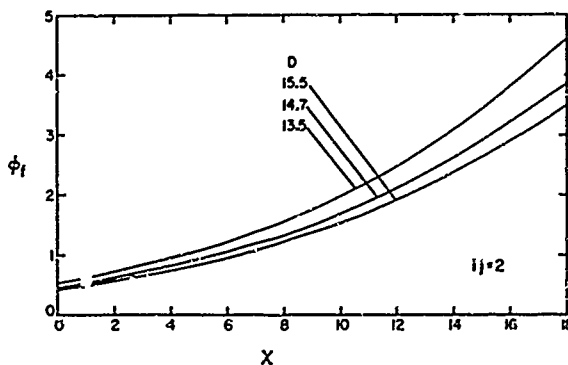


Figure 8. Frozen Vibrational Temperature Function ϕ_f with Similarity Parameter X for Nitrogen

Parameter ξ

The parameter ξ is defined as

$$\xi = S_0 - \log_e \left[\frac{uA}{\rho_* u_*} \right] \quad (36)$$

This parameter is not only a function of the reservoir and nozzle throat conditions but also a function of velocity, which is unknown. Therefore, the velocity ratio u'/u_* was computed for several sets of reservoir conditions and its variation with nozzle area ratio is shown in Fig 9. It is observed that this type of correlation still results in a significant amount of reservoir temperature effect which is similar to that noticed in the correlation of $(N_g)_1$ shown in Fig 2. Therefore, each velocity ratio was multiplied by the corresponding mass flow ratio and replotted in Fig 10. All the computed values now correlate very well and a mean curve can be drawn through the points as shown. This mean curve can be represented by a simple expression of the type

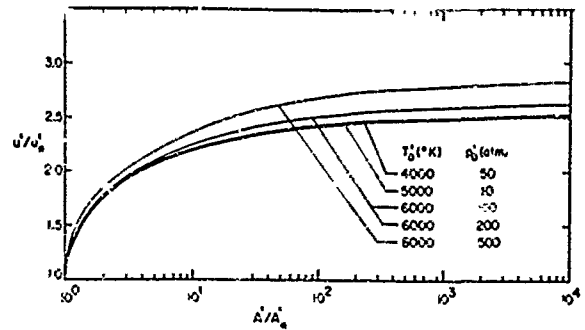


Figure 9. Velocity Ratio Variation with Area Ratio for Nitrogen

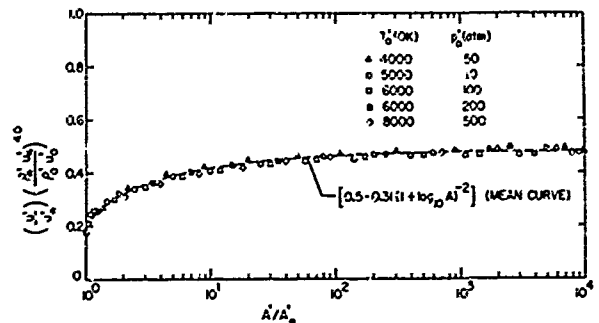


Figure 10. Velocity Function Correlation with Area Ratio for Nitrogen

$$\frac{u'}{u_*} \equiv \frac{u}{u_*} = \left(\frac{\rho_0^0 u_0^0}{\rho_*^0 u_*^0} \right)^{4.0} \times [0.5 - 0.31(1 + \log_{10} A)^{-2}] \quad (37)$$

The exponent 4.0 in Eq (37) was determined by using considerations similar to those used previously in the correlation of N_g . By substituting for S_0 , u'/u_* and ρ_*^0 from Eqs (21), (37) and (B2), respectively, an expression for ξ , after some algebraic manipulations, may be given as

$$\xi = \frac{\psi_0}{e^{\psi_0-1}} - \log_e \left\{ k_1^{-4} k_2^{-1} \psi_0^{5/2} (1 - e^{-\psi_0}) \times A[0.5 - 0.31(1 + \log_{10} A)^{-2}] \right\} + S_r \quad (38)$$

It is noted that ξ depends on only two parameters, namely, the area ratio A and the reservoir temperature function ψ_0 . The parameter ξ does not depend on either the reference density ρ_d or the reservoir density ρ_0^0 . The parameter ξ is now expressed in terms of the initial and boundary values only and hence can be readily computed for any given conditions.

Parameter χ

The parameter χ is defined as

$$\chi = \lambda - (1-1/ij) S_0 \quad (39)$$

Letting $\eta_0 = \log_e(\rho_0'/\rho_d)$ and expressing ρ_0' in terms of p_0' and ψ_0 from the equation of state, the expression for entropy (Eq (21)) is shown as

$$S_0 = \log_e \left[\frac{\rho_d R \theta_v \psi_0^{-7/2}}{p_0' m (1-e^{-\psi_0})} \right] + \frac{\psi_0}{e^{\psi_0}-1} + S_r \quad (40)$$

The parameter λ given in Eq (34) can be expressed as

$$\lambda = \log_e \left[\frac{k_1 (6+1/ij)}{ij C} \left(\frac{k \theta_v}{m} \right)^{(1/2-1/ij)} \times p_0'^{1/ij} \psi_0^{(1/2+1/ij)} L p_d^{(1-1/ij)} \right] \quad (41)$$

where the mass flow correlation expression given in Appendix B has been used to eliminate the mass flow term in λ . With Eqs (40) and (41), the parameter χ reduces to

$$\chi = \log_e \left[k_1 (6+1/ij) \left(\frac{m}{R \theta_v} \right)^{1/2} \left(\frac{p_0'}{C} \right) L \psi_0^{(4-2.5/ij)} (1-e^{-\psi_0})^{(1-1/ij)} \right] - (1-1/ij) \left[\frac{\psi_0}{e^{\psi_0}-1} + S_r \right] \quad (42)$$

It is noted that the terms $(R \theta_v/m)^{1/2}$, L and C/p_0' have the dimensions of velocity, length and time, respectively. ψ_0 and S_r are non-dimensional numbers. If p_0' is measured in atmospheres then the C units will be atm-sec. The parameter χ is independent of λ . For a given gas χ depends on only p_0' , L and ψ_0 since θ_v , C and S_r are all constant. The functions ϕ and ψ as well as the general parameters ξ and χ do not depend on any of the reference values that are used for nondimensionalizing the governing equations. Hence, the reference values can be chosen arbitrarily.

In Ref 6 the parameter $p_0' L$ was used as a correlating parameter. Also, in Ref 7 a parameter $(p_0' L \psi_0^{1/2}/C)$ was deduced by nondimensionalizing the sudden freezing criteria with reservoir values. These two parameters can be deduced from the present general parameter χ . To obtain the parameter $p_0' L$ the entire effect of ψ_0 has to be neglected. The other parameter can be obtained from χ by neglecting a portion of the effect of ψ_0 . This neglect of the ψ_0

effect is the reason why the frozen vibrational temperature presented in these analyses^{6,7} depends on the reservoir temperature in addition to the approximate parameters. Furthermore, these analyses predict only the approximate frozen vibrational temperature and do not provide the flow quantities in the nonequilibrium region.

Range of the Applicability of the Parameter χ

The general correlating parameter χ , as noted before, depends on p_0' , L and ψ_0 for a given gas. The variation of $p_0' L$ with ψ_0 for a constant χ can be computed from Eq (42) and is shown in Fig 11 for a number of χ values with $ij = 2.0$. The variation of p_0' ($L = 1.0$) with ψ_0 for a constant equilibrium mole fraction of 0.1 is also shown; this curve represents approximately the high temperature limit beyond which dissociation becomes appreciable and the dissociation relaxation may have to be considered in addition to the vibrational relaxation phenomenon. It can be inferred from Fig 11 that all of the practically feasible reservoir conditions (a maximum pressure of 100% atmospheres and a temperature range from 2000°K to 8000°K) are covered by χ values between 1 to 8, which is rather a narrow range compared to the range of χ values presented in Fig 5. It is also observed in Fig 5 that the nonequilibrium solution, for a given χ , departs from the equilibrium solution at a certain maximum value of ψ ; these maximum ψ values are also plotted in Fig 11. For reservoir conditions, which fall above this line, the solutions start with equilibrium conditions and can be obtained from the present similar solutions. For reservoir conditions which fall well below this line the flows can be taken as completely frozen in the entire nozzle. In a narrow region just below the equilibrium limit line the flow will be in the nonequilibrium state and the

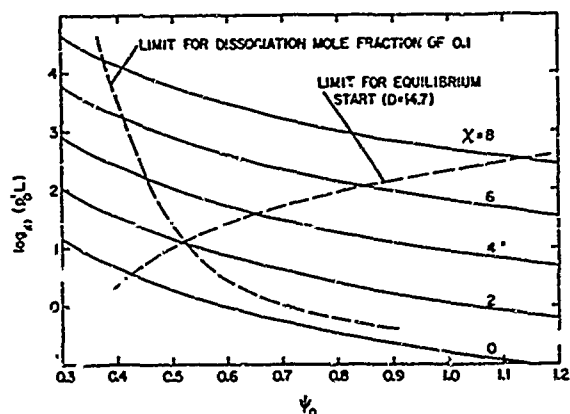


Figure 11. Range of Applicability of the Parameter χ ($ij = 2.0$, $C = 1.715 \times 10^{-11}$ atm-sec., p_0' in atm., L in cm)

solutions have to be obtained by starting with reservoir conditions as the initial values. Furthermore, the function N_g upstream of the throat has to be also included in the solutions.

Effect of Starting Value ξ_* on Similar Solutions

It is observed in Fig 5 that, for a given value of χ , there is a range of ψ_* (hence ξ_*) values that could be used as the initial starting values. The upper limit is given by the equilibrium limit line in Fig 11 and the lower limit is given by the dissociation limit as shown in Fig 11. For example, the range of ψ_* values is approximately 1.1 to 0.6 for $\chi = 8.0$ and 0.65 to 0.45 for $\chi = 4.0$. Hence, for a given χ , the nonequilibrium solutions could be started with any of a range of ξ_* values. This would result in slightly different values of N_g and, therefore, different nonequilibrium solutions for the same value of χ . To examine this point more closely, nonequilibrium solutions were obtained for different ξ_* values but with the same χ value and it was found that, for the range of allowable starting values mentioned previously, the differences in the nonequilibrium solutions varied only a few percent and this error is within the accuracy of the correlation of N_g shown in Fig 3.

VI. CONCLUSIONS

Based on the present analysis the following conclusions are reached:

1. Similar solutions for vibrational nonequilibrium nozzle flow problems can be obtained by using the new similarity parameter ξ .
2. The similar solutions presented can be used over a wide range of practicable combinations of initial conditions and nozzle scale parameters.
3. The vibrational equilibrium solutions depend on the one parameter ξ only and the nonequilibrium solutions depend on two parameters ξ and χ . The frozen vibrational temperature depends on χ only.
4. The parameters ξ and χ serve as universal correlating parameters since they contain all the parameters of the problem.
5. The present similarity transformation of the governing equations affords a better insight of the parametric dependence in this problem and should be of interest to theoreticians as well as experimentalists.

APPENDIX A

The Cut-Off Harmonic Oscillator Approximation

The simple harmonic oscillator model assumes the diatomic molecule may be

vibrationally excited through an infinite number of equally spaced energy levels. In the real situation, the excited molecules dissociate when the vibration energy level corresponding to the dissociation energy is reached. Therefore, in the cut-off harmonic oscillator approximation the vibration energy is considered only up to the dissociation limit. Since the energy levels are assumed to be equally spaced, the number N of allowable energy levels, when applying this approximation is given by $N = T_d/\theta_v$.

The factors which take into account the effect of the cut-off harmonic oscillator approximation are given below:

$$f_1 = \left[\frac{N^2 e^{N\phi} (e^\phi - 1)^2}{e^\phi (e^{N\phi} - 1)^2} \right] \quad (A1)$$

$$f_2 = \left[\frac{N(e^\phi - 1)(e^\psi - 1)(e^{N\phi} - e^{N\psi})}{(e^\phi - e^\psi)(e^{N\phi} - 1)(e^{N\psi} - 1)} \right] \quad (A2)$$

$$f_3 = \left[\log_e(1 - e^{-N\phi}) - \frac{N\psi}{(e^{N\phi} - 1)} \right] \quad (A3)$$

It can be easily shown that these factors tend to zero as $N \rightarrow \infty$. This limit corresponds to the simple harmonic oscillator approximation. Since no other expression in the governing equations contains the parameter N , the governing equations for a simple harmonic oscillator approximation can be obtained by simply assuming these factors are equal to zero. Furthermore, it can be shown that for nitrogen, with $N = 34$, the correction factors are all very small even for temperatures as high as 7000°K. For example, at $T_t = T_v = 6000^\circ\text{K}$ and $N = 34$, $f_1 = 2.28 \times 10^{-6}$, $f_2 = 0$, $f_3 = -8.7 \times 10^{-8}$. Thus, for the problem under consideration, the simple harmonic oscillator model is more than adequate.

APPENDIX B

Correlation of Mass Flow and Throat Density

The general correlating parameters ξ and χ contain the mass flow $\rho_* u_*$ and critical throat density ρ_* which have to be obtained in a separate computation. If the flow at the throat is in nonequilibrium, then the quantities $\rho_* u_*$ and ρ_* have to be computed by a trial and error procedure. However, in the present analysis the flow is considered to be in vibrational equilibrium up to the throat for which situation the mass flow and the throat density can be obtained by a set of algebraic equations. In the present case, the mass flow values were computed for nitrogen over a wide range of reservoir conditions by using the computer

program of Ref 12 and are presented in Fig B1. The nondimensional mass flow $(\rho_* u_*)/(\rho_0^* u_0^*)$ is independent of reservoir pressure but slightly dependent on reservoir temperature. It can be represented by a linear equation of the type

$$\rho_* u_* = k_1 (\rho_0^* u_0^*) \quad (B1)$$

where

$$k_1 = [0.689 - 6.3 \times 10^{-6} T_0^* (\text{°K})]$$

A similar correlation for the nondimensional throat density ρ_*/ρ_0^* is also shown in Fig B1. It is also independent of reservoir pressure and can be represented by a linear equation of the type

$$\rho_* = k_2 \rho_0^* \quad (B2)$$

where

$$k_2 = [0.634 - 2.33 \times 10^{-6} T_0^* (\text{°K})]$$

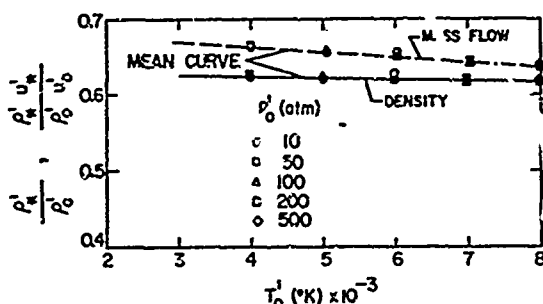


Figure B1. Correlation of Throat Density and Mass Flow Ratios for Nitrogen

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<p>The problem of obtaining similar solutions in pseudo-one-dimensional nozzle flows with vibrational relaxation is considered. The governing equations are transformed into a similar form by using a new similarity parameter η so that the nonequilibrium flow solutions depend on two parameters for a given kind of gas. However, the similar equations are further reduced to a universal form by a transformation of the independent variable η to ξ such that the similar solutions, for all combinations of initial conditions and for a defined family of nozzle shapes, depend on a single parameter χ which are then presented in a single graph. It is also shown that the equilibrium solutions depend on only one parameter ξ. The parameters ξ and χ are also the exact, general correlating parameters. With the present similar solutions the flow quantities in a nozzle are readily determined without resorting to complex computer programs.</p>			

KEY WORDS

nozzle flows

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